

# Three-body bound states in atomic mixtures with resonant $p$ -wave interaction

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We employ the Born-Oppenheimer approximation to find the effective potential in a three-body system consisting of a light particle and two heavy ones when the heavy-light short-range interaction potential has a resonance corresponding to a non-zero orbital angular momentum. In the case of an exact resonance in the  $p$ -wave scattering amplitude, the effective potential is attractive and long-range, namely it decreases as the third power of the inter-atomic distance. Moreover, we show that the range and power of the potential, as well as the number of bound states are determined by the mass ratio of the particles and the parameters of the heavy-light short-range potential.

*Introduction.* One of the most intriguing phenomenon of few-body physics is the Efimov effect [1], which manifests itself in an infinite number of weakly bound three-body states if at least two of the three two-body subsystems exhibit a single weakly  $s$ -wave bound state or resonance. The underlying effective potential is attractive and decreases as the *second* power of the inter-atomic distance [1]. In this Letter, we consider a three-body system consisting of a light particle and two heavy ones, when the heavy-light short-range interaction potential has a weakly bound or quasi-bound, *i.e.*, resonant state with a non-zero orbital angular momentum. We show that in the case of the exact  $p$ -wave resonance, the effective potential is also attractive and of long-range, but now decreases as the *third* power of the inter-atomic distance.

The Efimov effect occurs in systems where the two-body scattering length  $a_0$  is large compared to the characteristic radius  $R_0$  of the two-body interaction and displays an universal behavior, that is the details of the underlying short-range physics become irrelevant. In the resonant limit, *i.e.*,  $|a_0| \rightarrow \infty$ , the energies  $E_n$  of the three-body states form a geometric sequence, with the common ratio determined by the exponent  $\exp(2\pi/s_0)$ , that is  $\ln(E_n/E_{n+1}) = (2\pi/s_0)$  with  $n = 1, 2, 3, \dots$ . The parameter  $s_0$  depends on the masses of the particles and the number of participating resonant two-body interactions (two or three).

Examples of systems with a large scattering length are halo nuclei [2, 3] and the helium trimer [4]. In both cases  $a_0$  is exceedingly large, but not tunable. However, in order to observe the Efimov spectrum, it is crucial to be able to tune  $a_0$ . In the domain of ultracold atomic gases this task is achieved by Feshbach resonances [5] and different features of the three-body recombination process as well as the scattering of the atom off the shallow dimer, which are associated with the Efimov effect, have been measured [6] in this way. An additional prerequisite to detect many Efimov states is the use of an atomic mixture [7] with heavy atoms of mass  $M$  and light ones of mass  $m$ , since in this case the ratio of two nearest bound-state

energies is  $|E_{n+1}/E_n| \simeq 1$  for  $M/m \rightarrow \infty$  [8–10].

In our Letter we consider such an atomic mixture. However, in contrast to the standard Efimov scenario we focus on an exact  $p$ -wave resonance in the heavy-light short-range potential and determine the effective interaction potential between the two heavy atoms. Moreover, we demonstrate that the spectrum of bound states is solely determined by the mass ratio of the heavy and light particles, and the width of the  $p$ -wave state.

*Born-Oppenheimer approach.* Our three-body system consists of a light particle which interacts with two heavy particles and therefore can be easily analyzed within the Born-Oppenheimer approximation [11]. For this reason, the Schrödinger equation for the full wave function  $\Phi(\mathbf{r}, \mathbf{R}) = \Psi(\mathbf{r}; \mathbf{R})\chi(\mathbf{R})$  separates into two equations and the one for the light particle reads

$$\left[ -\frac{\hbar^2}{2\mu} \Delta_{\mathbf{r}} + U(\mathbf{r}_-) + U(\mathbf{r}_+) \right] \Psi(\mathbf{r}; \mathbf{R}) = -\frac{\hbar^2 \kappa^2}{2\mu} \Psi(\mathbf{r}; \mathbf{R}) \quad (1)$$

with  $\mathbf{r}_{\pm} \equiv \mathbf{r} \pm \frac{1}{2}\mathbf{R}$ . Here  $\mu \equiv 2mM/(2M + m) \approx m$  and  $\mathbf{R}$  denote the reduced mass and the separation between the two heavy particles, respectively. For the sake of simplicity we assume the potential  $U$  to be spherically symmetric, *i.e.*,  $U(\mathbf{r}) = U(r)$ , and to have the finite range  $R_0$ , *i.e.*,  $U(r > R_0) = 0$ .

The bound-state energy

$$\mathcal{V}(\mathbf{R}) \equiv -\frac{[\hbar\kappa(\mathbf{R})]^2}{2\mu} \quad (2)$$

of the light particle serves as an interaction potential for the relative motion of the heavy particles given by

$$\left\{ \Delta_{\mathbf{R}} + \frac{M}{\hbar^2} [E - \mathcal{V}(\mathbf{R})] \right\} \chi(\mathbf{R}) = 0, \quad (3)$$

with  $E$  being the total three-body energy.

*Interaction potential from scattering approach.* Next we determine  $\mathcal{V}$  by a self-consistent scattering of the light particle off the two heavy ones. For this purpose, we cast

Eq. (1) into the integral equation [12]

$$\Psi(\mathbf{r}) = -\frac{\mu}{2\pi\hbar^2} \int d\mathbf{r}' [U(\mathbf{r}'_-) + U(\mathbf{r}'_+)] \Psi(\mathbf{r}') \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \quad (4)$$

Since the total heavy-light potential  $U(\mathbf{r}_-) + U(\mathbf{r}_+)$  is nonzero only inside two spheres of radius  $R_0$  centered at  $\mathbf{r} = \pm \frac{1}{2}\mathbf{R}$ , we represent Eq. (4) as the superposition

$$\Psi(\mathbf{r}) = \Psi^{(-)}(\mathbf{r}) + \Psi^{(+)}(\mathbf{r}) \quad (5)$$

of the two waves

$$\Psi^{(\pm)}(\mathbf{r}) \equiv \int_{|\mathbf{r}' \pm \frac{\mathbf{R}}{2}| \leq R_0} d\mathbf{r}' \sigma^{(\pm)}(\mathbf{r}') \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \quad (6)$$

with

$$\sigma^{(\pm)}(\mathbf{r}) \equiv -\frac{\mu}{2\pi\hbar^2} U\left(\mathbf{r} \pm \frac{1}{2}\mathbf{R}\right) \Psi(\mathbf{r}). \quad (7)$$

The expansion

$$\frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{\kappa|\mathbf{r}-\mathbf{r}'|} = 8 \sum_{l=0}^{\infty} \sum_{|m_l| \leq l} \mathcal{I}_l(\kappa r') \mathcal{K}_l(\kappa r) Y_{lm_l}(\mathbf{n}_{\mathbf{r}'}) Y_{lm_l}(\mathbf{n}_{\mathbf{r}}) \quad (8)$$

into the modified spherical Bessel functions  $\mathcal{I}_l(z) \equiv \sqrt{\pi/(2z)} I_{l+1/2}(z)$  and  $\mathcal{K}_l(z) \equiv \sqrt{\pi/(2z)} K_{l+1/2}(z)$  [13], which is valid for  $r > r'$ , transforms Eq. (6) into

$$\Psi^{(\pm)}(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{|m_l| \leq l} C_{lm_l}^{(\pm)} \mathcal{K}_l(\kappa r_{\pm}) Y_{lm_l}(\mathbf{n}_{\mathbf{r}_{\pm}}). \quad (9)$$

Here  $Y_{lm_l}(\mathbf{n}_{\mathbf{r}}) \equiv Y_{lm_l}(\theta_r, \varphi_r)$  are the spherical harmonics with  $\mathbf{n}_{\mathbf{r}} \equiv \mathbf{r}/r = (\theta_r, \varphi_r)$ .

We regard the coefficients  $C_{lm_l}^{(\pm)}$  determined by the integral in Eq. (6) as independent variables and apply scattering theory to obtain from Eq. (9) explicit equations for  $C_{lm_l}^{(\pm)}$  coupled by the  $S$ -matrix elements of the potential  $U$ . For this purpose we consider a vicinity of the first potential well, that is  $\mathbf{r} = -\frac{1}{2}\mathbf{R} + \mathbf{x}$  with  $|\mathbf{x}| \approx R_0$ , where the total solution

$$\Psi\left(-\frac{\mathbf{R}}{2} + \mathbf{x}\right) = \sum_{l=0}^{\infty} \sum_{|m_l| \leq l} R_{lm_l}(\kappa|\mathbf{x}|) Y_{lm_l}(\mathbf{n}_{\mathbf{x}}) \quad (10)$$

given by Eq. (5) can be expanded into the spherical harmonics. Here the radial wave function

$$R_{lm_l}(\kappa|\mathbf{x}|) = C_{lm_l}^{(+)} \mathcal{K}_l(\kappa|\mathbf{x}|) + \pi(-1)^l \mathcal{I}_l(\kappa|\mathbf{x}|) \sum_{l'=0}^{\infty} \mathfrak{A}_{ll'}^{(m_l)} C_{l'm_l}^{(-)} \quad (11)$$

is determined by the sum of the two contributions resulting from  $\Psi^{(\pm)}(-\frac{1}{2}\mathbf{R} + \mathbf{x})$  defined by Eq. (9), and the

coefficients

$$\mathfrak{A}_{ll'}^{(m_l)}(\kappa R) \equiv \frac{1}{\pi} \sqrt{\frac{2l+1}{2l'+1}} \sum_{L=0}^{\infty} (-1)^L (2L+1) \times C_{l0L0}^{l'0} C_{lm_lL0}^{l'm_l} \mathcal{K}_L(\kappa R) \quad (12)$$

originate from the re-expansion [14] of  $\mathcal{K}_l(\kappa|\mathbf{x} - \mathbf{R}|) Y_{lm_l}(\mathbf{n}_{\mathbf{x}-\mathbf{R}})$  into  $Y_{lm_l}(\mathbf{n}_{\mathbf{x}})$  with the Clebsch-Gordan coefficients  $C_{lm_lL0}^{l'm_l}$ .

In order to derive an equation for  $C_{lm_l}^{(\pm)}$  we cast the radial wave  $R_{lm_l}$  given by Eq. (11) into the superposition

$$R_{lm_l}(\kappa|\mathbf{x}|) = a_l(\kappa) h_l^{(1)}(i\kappa|\mathbf{x}|) + b_l(\kappa) h_l^{(2)}(i\kappa|\mathbf{x}|) \quad (13)$$

of *outgoing* and *incoming* radial waves  $h_l^{(1)}$  and  $h_l^{(2)}$  with amplitudes

$$a_l(\kappa) = -\frac{\pi i^l}{2} C_{lm_l}^{(+)} + \frac{\pi i^l}{2} \sum_{l'=0}^{\infty} \mathfrak{A}_{ll'}^{(m_l)}(\kappa R) C_{l'm_l}^{(-)} \quad (14)$$

and

$$b_l(\kappa) = \frac{\pi i^l}{2} \sum_{l'=0}^{\infty} \mathfrak{A}_{ll'}^{(m_l)}(\kappa R) C_{l'm_l}^{(-)}. \quad (15)$$

The spherical Bessel functions of the third kind  $h_l^{(1)}$  and  $h_l^{(2)}$  are determined [13] in terms of  $\mathcal{K}_l$  and  $\mathcal{I}_l$  as  $\mathcal{K}_l(z) = -(\pi i^l/2) h_l^{(1)}(iz)$  and  $\mathcal{I}_l(z) = [h_l^{(1)}(iz) + h_l^{(2)}(iz)]/(2i^l)$ .

Since the amplitudes  $a_l$  and  $b_l$  of the *outgoing* and *incoming* waves are coupled [11, 15] by the  $S$ -matrix elements  $S_l$  of the scattering potential  $U$ , that is

$$a_l(\kappa) = S_l(i\kappa) b_l(\kappa), \quad (16)$$

we arrive at

$$C_{lm_l}^{(+)} + [S_l(i\kappa) - 1] \sum_{l'=0}^{\infty} \mathfrak{A}_{ll'}^{(m_l)}(\kappa R) C_{l'm_l}^{(-)} = 0. \quad (17)$$

Similarly we obtain from the second potential well, centered at  $\mathbf{r} = \frac{1}{2}\mathbf{R}$ , the relation

$$C_{lm_l}^{(-)} + [S_l(i\kappa) - 1] \sum_{l'=0}^{\infty} (-1)^{l+l'} \mathfrak{A}_{ll'}^{(m_l)}(\kappa R) C_{l'm_l}^{(+)} = 0. \quad (18)$$

Equations (17) and (18) constitute a system of linear algebraic equations for  $C_{lm_l}^{(\pm)}$  determining via Eq. (9) the waves  $\Psi^{(\pm)}$ . Its solution is nonzero only if the corresponding determinant vanishes which provides us with a transcendental equation for  $\kappa = \kappa(R)$ , and thus for the interaction potential  $\mathcal{V}$  defined by Eq. (2). The coefficients of these equations are determined by the  $S$ -matrix elements of the interaction potential  $U$  between the heavy and the light atoms.

*Zero-range limit.* In order to test our method, we first consider a zero-range potential, for which only  $s$ -wave scattering occurs and the  $S$ -matrix elements read [15, 16]

$$S_l(i\kappa) - 1 = \frac{2\kappa}{1/a_0 - \kappa} \delta_{l,0}. \quad (19)$$

In this case, the system Eqs. (17) and (18) reduces to two algebraic equations for  $C_{00}^{(\pm)}$  and has non-trivial solutions only if

$$[S_0(i\kappa) - 1] \mathfrak{A}_{00}^{(0)}(\kappa R) = \pm 1, \quad (20)$$

with  $\mathfrak{A}_{00}^{(0)}(\kappa R) \equiv [1/(2\kappa R)]e^{-\kappa R}$  defined by Eq. (12). This condition translates into equation

$$\frac{1}{\xi - \alpha_0 \rho} e^{-\xi} = \pm 1 \quad (21)$$

for  $\xi \equiv \kappa R$  with the parameters

$$\alpha_0 \equiv \frac{R_0}{a_0} \quad \text{and} \quad \rho \equiv \frac{R}{R_0}, \quad (22)$$

and coincides with the equation for the bound-state energy obtained in Refs. [9, 10, 16] for the case of the zero-range potential.

In the case of a  $s$ -wave resonance, that is  $\alpha_0 = 0$ , Eq. (21) has a solution  $\xi = \xi_* \approx 0.57$  only for the plus sign on the right-hand side, which translates into the familiar Efimov potential

$$\mathcal{V}^{(0)} \equiv -\frac{[\hbar\kappa_+(R)]^2}{2\mu} = -\frac{\hbar^2}{2\mu} \frac{\xi_*^2}{R^2}, \quad (23)$$

decaying with the second power of  $R$ .

*P-wave resonance.* Next we focus on the low-energy limit, that is on  $|E| \ll \hbar^2/(\mu R_0^2)$ , or  $\kappa R_0 \ll 1$ . In this case the  $S$ -matrix elements in each partial wave with the orbital angular momentum  $l$  can be presented [18, 19] in the form of the effective-range expansion

$$S_l(i\kappa) - 1 = (-1)^l \frac{2\kappa^{2l+1}}{1/a_l - (r_l/2)\kappa^2 + (-1)^{l+1}\kappa^{2l+1}}. \quad (24)$$

The resonance regime in the  $l$ -th partial wave is reached when the absolute value of the *effective scattering length*  $|a_l| \gg R_0^{2l+1}$ . The effective range  $r_l$  is positive for  $l > 0$  and linked [5, 20, 21] to the width of the resonance.

We now consider the case of a resonant  $p$ -wave, that is a partial wave with  $l = 1$ , and substitute the  $S$ -matrix elements given by Eq. (24) into Eqs. (17) and (18). Since  $S_0(i\kappa) - 1 \sim S_1(i\kappa) - 1 \sim \kappa R_0$  and  $S_{l>1}(i\kappa) - 1 \sim (\kappa R_0)^{2l+1}$ , we neglect the small terms with  $l = 2, 3, \dots$  and arrive at two separate systems of equations with respect to the projection  $m_l$  of the angular momentum.

For  $m_l = \pm 1$ , Eqs. (17) and (18) simplify to two equations for  $C_{1,\pm 1}^{(\pm)}$  and have a non-trivial solution only if

$$[S_1(i\kappa) - 1] \mathfrak{A}_{11}^{(\pm 1)}(\kappa R) = \pm 1. \quad (25)$$

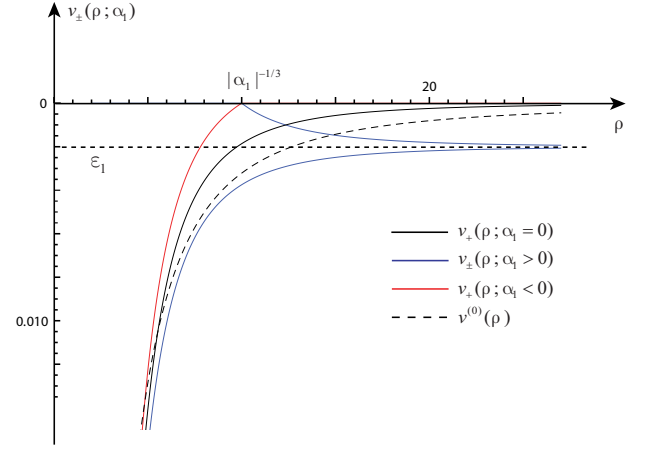


Figure 1: (Color online) Effective potentials  $\mathcal{V}_{\pm}^{(1,\pm 1)}(\rho; \alpha_1) \equiv -[\hbar^2/(2\mu R_0^2)]v_{\pm}(\rho; \alpha_1)$  between two heavy particles induced by  $p$ -wave resonant scattering of a light particle on both sides of the corresponding Feshbach resonance. Here  $v_{\pm} = (\xi_{\pm}/\rho)^2$  are determined by the solutions  $\xi_{\pm}(\rho; \alpha_1)$  of Eq. (26) for  $|\alpha_1| = 10^{-3}$ ,  $\beta = 0.5$ , and  $\alpha_1 = 0$ ;  $\varepsilon_1 = -(\alpha_1/\beta)$  is the bound-state energy of the light atom in the potential  $U$ . The dashed line describes the familiar Efimov potential  $v^{(0)} \equiv -(\xi_*/\rho)^2$  defined by Eq. (23).

Since  $S_1(i\kappa)$  is given by Eq. (24) and  $\mathfrak{A}_{11}^{(\pm 1)}(\kappa R) \equiv -\frac{3}{2}[(1 + \kappa R)/(\kappa R)^3]e^{-\kappa R}$ , Eq. (25) for  $\xi = \kappa^{(1)}R$  and distances  $R \geq 2R_0$  reads

$$\frac{(1 + \xi)}{\beta \rho \xi^2 - \alpha_1 \rho^3 - \frac{1}{3}\xi^3} e^{-\xi} = \pm 1 \quad (26)$$

with the dimensionless parameters

$$\alpha_1 \equiv \frac{R_0^3}{3a_1} \quad \text{and} \quad \beta \equiv \frac{r_1 R_0}{6}. \quad (27)$$

In the resonant case, that is  $\alpha_1 = 0$ , Eq. (26) has a solution only for the plus sign on the right-hand side. In the limit of  $0 < \xi \ll 1$ , we find  $\xi \cong (\beta \rho)^{-\frac{1}{2}}$  for  $\rho \gg 1$ , which translates into the potential

$$\mathcal{V}^{(1,\pm 1)} \equiv -\frac{[\hbar\kappa_{\pm}^{(1)}(R; 0)]^2}{2\mu} \cong -\frac{\hbar^2}{2\mu} \frac{6}{r_1 R^3}, \quad (28)$$

which is independent of  $R_0$ .

The potentials  $\mathcal{V}_{\pm}^{(1,\pm 1)}(R; \alpha_1) \equiv -[\hbar^2/(2\mu R_0^2)](\xi_{\pm}/\rho)^2$  determined by the solutions  $\xi_{\pm}(\rho; \alpha_1)$  of Eq. (26) are presented in Fig. 1 for  $|\alpha_1| = 10^{-3}$  and  $\beta = 0.5$ , and  $\alpha_1 = 0$ . The form of  $\mathcal{V}_{\pm}^{(1,\pm 1)}$  is determined by the sign of  $a_1$ , that is by the sign of  $\alpha_1$ , Eq. (27). Indeed, for  $\alpha_1 > 0$ , *i.e.*, in the case of the weakly-bound  $p$ -wave state in  $U$ ,  $\mathcal{V}_{+}^{(1,\pm 1)}$  as well as  $\mathcal{V}_{-}^{(1,\pm 1)}$  approach for large distances,  $R > R_0|\alpha_1|^{-1/3}$ , the bound state energy  $\varepsilon_1 \equiv -(\alpha_1/\beta)$  of the light particle. For short distances,  $R < R_0|\alpha_1|^{-1/3}$ ,  $\mathcal{V}_{\pm}^{(1,\pm 1)}(R; \pm|\alpha_1|)$  approach  $\mathcal{V}_{+}^{(1,\pm 1)}(R; \alpha_1 = 0) = \mathcal{V}^{(1,\pm 1)}(R)$ .

In the case of a  $p$ -wave resonance, the matrix element  $S_1$  corresponding to the resonant channel is of the same order as  $S_0$  for the non-resonant channel [11]. Therefore, for  $m_l = 0$  we have to take into account in Eqs. (17) and (18) both the  $s$ - and  $p$ -waves, which gives rise to a system of four algebraic equations for  $C_{0,0}^{(\pm)}$  and  $C_{1,0}^{(\pm)}$ , leading us to the relation

$$1 - (S_0 - 1)(S_1 - 1) \left( \mathfrak{A}_{00}^{(0)} \mathfrak{A}_{11}^{(0)} - \mathfrak{A}_{01}^{(0)} \mathfrak{A}_{10}^{(0)} \right) = \mp \left[ (S_0 - 1) \mathfrak{A}_{00}^{(0)} - (S_1 - 1) \mathfrak{A}_{11}^{(0)} \right]. \quad (29)$$

According to Eqs. (12), (24) and (27), we obtain  $\mathfrak{A}_{01}^{(0)} \equiv \mathfrak{A}_{10}^{(0)} = -\frac{\sqrt{3}}{2}[(1 + \kappa R)/(\kappa R)^2]e^{-\kappa R}$  and  $\mathfrak{A}_{11}^{(0)} \equiv \frac{3}{2}[(\kappa^2 R^2 + 2\kappa R + 2)/(\kappa R)^3]e^{-\kappa R}$ , and Eq. (29) for  $\xi = \kappa^{(0)} R$  takes the form

$$1 + \frac{e^{-2\xi}}{(\xi - \alpha_0 \rho)(\beta \rho \xi^2 - \alpha_1 \rho^3 - \frac{1}{3}\xi^3)} = \pm e^{-\xi} \left[ \frac{(\xi^2 + 2\xi + 2)}{\beta \rho \xi^2 - \alpha_1 \rho^3 - \frac{1}{3}\xi^3} + \frac{1}{\xi - \alpha_0 \rho} \right]. \quad (30)$$

In the resonant case,  $\alpha_1 = 0$ , we have  $\alpha_0 \sim 1$  and Eq. (30) has a solution only for the plus sign on the right-hand side. In the limit of  $0 < \xi \ll 1$ , we find  $\xi \cong [2/(\beta \rho)]^{\frac{1}{2}}$  for  $\rho \gg 1$ , giving rise to the potential

$$\mathcal{V}^{(1,0)} \equiv -\frac{[\hbar \kappa_+^{(0)}(R; 0)]^2}{2\mu} \cong -\frac{\hbar^2}{\mu} \frac{6}{r_1 R^3}. \quad (31)$$

The potentials  $\mathcal{V}_{\pm}^{(1,0)}(R; \alpha_1) \equiv -[\hbar^2/(2\mu R_0^2)](\xi_{\pm}/\rho)^2$  determined by the solutions  $\xi_{\pm}(\rho; \alpha_1)$  of Eq. (30) at  $\alpha_0 \sim 1$  and  $\beta \sim 1$  are similar to  $\mathcal{V}_{\pm}^{(1,\pm 1)}$  of Fig. 1 with two qualitative differences: (i) the ranges of  $\mathcal{V}_{\pm}^{(1,\pm 1)}$  and  $\mathcal{V}_{\pm}^{(1,0)}$  are different and equal to  $R_1 \equiv R_0|\alpha_1|^{-1/3}$  and  $R_2 \equiv R_0(0.5|\alpha_1|)^{-1/3}$ , respectively, and (ii) in the case of exact resonance,  $\mathcal{V}^{(1,0)}$  given by Eq. (31) has the same asymptotic behavior as  $\mathcal{V}^{(1,\pm 1)}$  defined by Eq. (28) with twice the amplitude.

*Spectrum of induced  $1/R^3$ -potential.* Finally we focus on the dynamics of the two heavy particles dictated by the Schrödinger equation (3) with the potential  $\mathcal{V}$  given by Eqs. (28) and (31) and induced by the  $p$ -wave resonance in the light-heavy interaction. We emphasize that  $\mathcal{V}$  is only meaningful for  $R \gg R_0$ , since for  $R \sim R_0$  it is likely to be determined by the inter-atomic forces and obviously cannot be treated by the effective-range expansion.

The energies  $E_n$  of the bound states with zero angular orbital momentum follow from the familiar WKB quantization rule [11]

$$n - n_0 = \frac{1}{\pi \hbar} \int_{R_0}^{R_{E_n}} \sqrt{M[E_n - \mathcal{V}(R)]} dR, \quad (32)$$

giving rise [22, 23] to the spectrum

$$E_n = -\frac{\hbar^2}{MR_*^2} \left( \frac{n_0 - n}{g} \right)^6 \quad (33)$$

for the weakly bound states. Here we have introduced the characteristic range

$$R_* \equiv (2 - |m_l|) \frac{3M}{\mu r_1} \quad (34)$$

of the effective potential  $\mathcal{V}$  in the resonant case, that is  $\alpha_1 = 0$ , and  $[n_0] - n = 1, 2, \dots$ , where the integer part  $[n_0]$ , determined by the phase of the wave function at the short distances  $R \sim R_0$ , plays a role of a three-body parameter and  $g = \Gamma(\frac{5}{6})/[\sqrt{\pi} \Gamma(\frac{4}{3})]$ .

Since  $\mathcal{V}$  defined by Eqs. (28) and (31) has a tail falling off faster than  $-1/R^2$ , it supports [11] only a finite number  $N_0$  of bound states with zero angular orbital moment. Indeed,  $N_0$  can be estimated [11] by the WKB method and yields

$$N_0 = \frac{1}{\pi} \int_{R_0}^{\infty} \sqrt{\frac{R_*}{R^3}} dR = \frac{2}{\pi} \sqrt{\frac{R_*}{R_0}} = \frac{2}{\pi} \left[ \frac{(2 - |m_l|) 3M}{r_1 R_0 m} \right]^{\frac{1}{2}}, \quad (35)$$

that is  $N_0$  is determined by the square root of the ratio of the mass-ratio  $M/m$  to the dimensionless effective range  $r_1 R_0$  of the  $p$ -wave resonance.

*Scattering off  $1/R^3$ -potential and summary.* The appearance of  $\mathcal{V}$  given by Eqs. (28) and (31) can be verified experimentally by scattering a heavy atom off the diatomic molecule consisting of a heavy and a light atom. The predicted three-body bound states manifest themselves as resonances in the cross-section of the atom-molecule scattering when we tune the magnetic field close to the  $p$ -wave Feshbach resonance. Moreover, due to the inverse-cube tail the cross-section  $\sigma_L$  of the  $L$ -th partial wave has the unique behavior [24],  $\sigma_0(E) = \pi R_*^2 \ln^2(MR_*^2 E/\hbar^2)$  and  $\sigma_{L>0}(E) = \pi R_*^2 [(2L+1)/(L^2+L)^2]$  at the low incident energy  $E \ll \hbar^2/(MR_*^2)$ .

In summary, we have found a novel series of bound states in the three-body system consisting of a light particle and two heavy ones when the heavy-light short-range interaction potential has the  $p$ -wave resonance. In the case of an exact resonance, the effective potential is attractive and of long-range. Moreover, the spectrum of bound states is determined by the mass ratio of the heavy and light particles as well as the parameters of the heavy-light short-range potential.

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